ON LOCAL INTEGRABILITY CONDITIONS OF JET GROUPOIDS

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Abstract. A Jet groupoid $\mathcal{R}_q$ over a manifold X is a special Lie groupoid consisting of q-jets of local diffeomorphisms $X \to X$. As a subbundle of $J_q(X \times X)$, a jet groupoid can be considered as a nonlinear system of partial differential equations (PDE). This leads to the concept of formal integrability. On the other hand, each jet groupoid is the symmetry groupoid of a geometric object, modelled as a section ω of a natural bundle $\mathcal{F}$. Using the jet groupoids, we give a local characterisation of formal integrability for transitive jet groupoids in terms of their corresponding geometric objects.

1. Introduction

In two articles [6], Lie introduced pseudogroups and formulated their defining equations as differential invarants of the corresponding pseudogroup. Vessiot continued in [13] with the calculation of necessary conditions for integrability. Later, Pommaret [9] applied Spencer's approach [10] to pseudogroups and showed that they are all given as symmetry transformations of geometric objects ω on natural bundles $\mathcal{F}$.

In this paper, we use the language of jet groupoids which provides an efficient language for this theory, including conceptional proofs. A Jet groupoid consists of the algebraic solutions of the pseudogroup equations and is thus also defined by a section of a natural bundle. We express the prolongation and projection of jet groupoids by sections of new natural bundles. All considerations are local. As the main result, we obtain conditions on the sections ω of $\mathcal{F}$ that classify all formally integrable jet groupoids that can be defined by sections of $\mathcal{F}$.

The last section gives explicit calculations for the well-known example of a Riemannian metric on a two-dimensional manifold using computer algebra.

2. Preliminaries

Before turning to jet groupoids, we shortly introduce the underlying Lie groupoids and their action on fibre bundles. This is done to fix the notation, recent introductory books on Lie groupoids are [7] and [8], which also provide many further references.

Definition 2.1. A groupoid G is a small category with invertible morphisms.

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The set of objects, denoted by $G^{(0)}$, is called the *base*. The set of morphisms is denoted by $G^{(1)}$ and has the projections *source* and *target* to the base:

$$s : G^{(1)} \to G^{(0)} : g \mapsto x \qquad t : G^{(1)} \to G^{(0)} : g \mapsto y \qquad \text{for } g : x \to y \in G^{(1)}.$$

Composition of morphisms induces a partial multiplication that is defined whenever source and target match:

$$\mu : G^{(1)} \, {}_{s}\!\underset{G^{(0)}}{\curlywedge}\!{}_{t} \, G^{(1)} \to G^{(1)} : (g, h) \mapsto gh$$

with $G^{(1)} \, {}_{s}\!\curlywedge^{t}_{G^{(0)}} G^{(1)} = \{(g, h) \in G^{(1)} \times G^{(1)} | s(g) = t(h)\}$. Via $\iota : G^{(0)} \hookrightarrow G^{(1)} : x \mapsto 1_x$, the base is embedded in $G^{(1)}$ as the identity morphisms. A groupoid is called *transitive* if $G(x, y) = \{g \in G^{(1)} | s(g) = x, t(g) = y\} \neq \emptyset$ for all $x, y \in G^{(0)}$.

Definition 2.2. A groupoid G is called a *Lie groupoid* if $G^{(0)}$ and $G^{(1)}$ are smooth[1] manifolds where s, t, μ, ι and the inversion are smooth, s, t being surjective submersions.

In the case of Lie groupoids, $G^{(1)} \, {}_{s}\!\curlywedge^{t}_{G^{(0)}} G^{(1)}$ turns into the fibre product and the isotropy groups $G(x, x)$ are Lie groups [8, Thm 5.4]. Important examples of Lie groupoids are gauge groupoids $PP^{-1} := P \times_H P$ for Lie groups H and principal H-bundles P.

Definition 2.3. Let G be a Lie groupoid and $\pi : \mathcal{F} \to G^{(0)}$ be a fibre bundle. A right *groupoid action* of G on $\mathcal{F}$ is a smooth map $\mathcal{F} \, {}_{\pi}\!\curlywedge^{t}_{G^{(0)}} G^{(1)} \to \mathcal{F}$ with $f(ab) = (fa)b$ and $f1_x = f$ whenever $f \in \mathcal{F}_x = \pi^{-1}(x)$ and $a, b \in G^{(1)}$ can be composed.

3. Jet Groupoids and Natural Bundles

We will now define jet groupoids as Lie groupoids over a fixed base manifold $G^{(0)} = X$ of dimension n. The q-th jet bundle $J_q(X \times X)$ over the trivial bundle $X \times X$ provides source $s = \mathrm{pr}_1$ and target $t = \mathrm{pr}_2$ as projections on the first and second copy of X. Restricted to the open subset $\Pi_q \subset J_q(X \times X)$ of invertible jets, the chain rule induces a partial multiplication on Π_q, which is respected by the natural projections $\pi^{q+r}_q : J_{q+r}(X \times X) \to J_q(X \times X)$.

Definition 3.1. Π_q is called the *full jet groupoid* of order q and a *jet groupoid* $\mathcal{R}_q$ is a subbundle of Π_q, closed with respect to all groupoid operations.

For the treatment of natural bundles and the projection of jet groupoids, it is helpful to consider the isotropy groups $\Pi_q(x, x)$. They are all isomorphic to $\mathrm{GL}_q = \mathrm{GL}_q(\mathbb{R}^n)$, the Lie group of q-jets of diffeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ leaving the origin fixed. The structure of GL_q was studied by Terng [12]. By the construction of GL_{q+1} there is an exact sequence

$$1 \longrightarrow K_{q+1} \longrightarrow \mathrm{GL}_{q+1}(\mathbb{R}^n) \xrightarrow{\pi^{q+1}_q} \mathrm{GL}_q(\mathbb{R}^n) \longrightarrow 1 \tag{3.1}$$

defining the normal subgroup $K_{q+1} := \ker(\pi^{q+1}_q) \trianglelefteq \mathrm{GL}_{q+1}$. The projection can be identified with $\pi^{q+1}_q : \Pi_{q+1}(x, x) \to \Pi_q(x, x)$ for each $x \in X$. GL_{q+1} is called first principal prolongation of GL_q in [4] (following [2]).

[1]In this paper, smooth means C^∞.

All jets with fixed target $y_0 \in X$ define a principal GL_q-bundle $P_q := \Pi_q(-, y_0)$. When changing from groupoids to the bundle point of view, left and right $\mathrm{GL}_q \cong \Pi_q(y_0, y_0)$-actions must be swapped to obtain the equations in [6], [13] and (right) principal bundles. We recover Π_q as the gauge groupoid $P_q P_q^{-1}$ (via $(g, h) \mapsto gh^{-1}$). The sequence (3.1) implies $P_{q+1}/K_{q+1} \cong P_q$. Writing K_{q+1} for all kernels $\ker(\pi_q^{q+1}) \trianglelefteq \Pi_{q+1}(x, x)$, we obtain a kind of commutation law $K_{q+1} f_{q+1} = f_{q+1} K_{q+1}$ as sets for all $f_{q+1} \in \Pi_{q+1}$.

Definition 3.2. A fibre bundle $\mathcal{F} \xrightarrow{\pi} X$ is called *natural bundle* if there exists a $q \in \mathbb{N}$ and a groupoid action of Π_q on $\mathcal{F}$. A section ω of $\mathcal{F}$ is called *geometric object*.

P_q is a natural bundle by right Π_q-multiplication. In fact, all natural bundles $\mathcal{F}$ with typical fibre $F := \mathcal{F}_{y_0}$ are associated to P_q as $\mathcal{F} \cong P_q \times_{\mathrm{GL}_q} F$. This is done by splitting $u \in \mathcal{F}$ into $u = u_{y_0} f_q$ with $u_{y_0} \in F$ and $f_q \in P_q$, unique up to elements of $\mathrm{GL}_q \cong \Pi_q(y_0, y_0)$.

If not stated otherwise, we assume the natural bundles $\mathcal{F}$ to have fibres F that are homogeneous GL_q-spaces.

Proposition 3.3. Each section ω of a natural bundle $\mathcal{F}$ defines a jet groupoid $\mathcal{R}_q(\omega) = \mathrm{Stab}^q_{\mathcal{F}}(\omega)$. Conversely, each transitive jet groupoid $\mathcal{R}_q$ defines a natural bundle $\mathcal{F}$ with section ω_0, such that $\mathcal{R}_q$ is the full symmetry groupoid $\mathrm{Stab}^q_{\mathcal{F}}(\omega_0)$ of ω_0.

Proof. Define the symmetry groupoid $\mathrm{Stab}^q_{\mathcal{F}}(\omega)$ via the Π_q-action on $\mathcal{F}$

$$\Phi_\omega : \Pi_q \to \mathcal{F} : f_q \mapsto \omega(t(f_q)) f_q$$

as the kernel $\ker_\omega(\Phi_\omega) = \{f_q \in \Pi_q | \Phi_\omega(f_q) = \omega(s(f_q))\}$ or by the exact sequence

$$0 \longrightarrow \mathrm{Stab}^q_{\mathcal{F}}(\omega) \longrightarrow \Pi_q \underset{\omega \circ s}{\overset{\Phi_\omega}{\rightrightarrows}} \mathcal{F} \tag{3.2}$$

of bundles over $X = s(\Pi_q)$. The Π_q-action implies that $\mathrm{Stab}^q_{\mathcal{F}}(\omega)$ is a groupoid since it is closed under μ, ι and inversion. As F is homogeneous, Φ_ω is surjective and of constant rank. By the implicit function theorem, $\mathrm{Stab}^q_{\mathcal{F}}(\omega)$ is a Lie groupoid. Each $f_q \in \Pi_q$ can be modified by $g_q \in \mathrm{GL}_q$ such that $\omega(y) f_q g_q = \omega(x)$, so $\mathrm{Stab}^q_{\mathcal{F}}(\omega)$ is transitive.

The transitivity of $\mathcal{R}_q$ implies that all isotropy groups are isomorphic to some $G_q \leq \mathrm{GL}_q$, so choose $y_0 \in X$ and set $\mathcal{F} := \mathcal{R}_q(y_0, y_0) \backslash \Pi_q(-, y_0) \cong P_q \times_{\mathrm{GL}_q} \mathrm{GL}_q / G_q$ with the section

$$\omega_0 : X \to \mathcal{F} : x \mapsto \mathcal{R}_q(y_0, y_0) \mathcal{R}_q(x, y_0) = \mathcal{R}_q(x, y_0).$$

The condition for $r_q \in \mathrm{Stab}^q_{\mathcal{F}}(\omega_0)$ is $\omega_0(y) r_q = \omega_0(y)$ or explicitly $\mathcal{R}_q(y, y_0) r_q = \mathcal{R}_q(x, y_0)$. This is equivalent to $r_q \in \mathcal{R}_q$. $\square$

The groupoids defined by different sections may be the same, e. g. if $\mathcal{F}$ is a vector bundle, ω and $\lambda\omega$ for a constant $\lambda \neq 0$ describe the same groupoid. Proposition 3.3 can be extended to non-homogeneous fibres F with the additional assumption that Φ_ω has constant rank ($\omega(y) 1_y = \omega(y)$ implies $\mathrm{im}(\omega) \subseteq \mathrm{im}(\Phi_\omega)$). Vessiot [13] calls the coordinate expressions of $\Phi_\omega(r_q) = \omega(s(r_q))$ Lie form. The sequence (3.2) is due to Pommaret [9].

4. Systems of PDE and Formal Integrability

The next step is to consider a jet groupoid as a system of PDE (see e. g. [3], [5], [10]) in order to study its integrability. The prolongation of a groupoid acting on a fibre bundle was introduced by Ehresmann [2] (see also [4]).

Definition 4.1. A subbundle $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ of the q-th order jet bundle of a fibre bundle $\mathcal{E} \to X$ is called *system of PDE* and solutions are (local) sections of $\mathcal{R}_q$. The *r-prolongation* is the subset

$$\mathcal{R}_{q+r} := J_r(\mathcal{R}_q) \cap J_{q+r}(\mathcal{E}), \qquad r \in \mathbb{Z}_{\geq 0}$$

and

$$\mathcal{R}^{(s)}_{q+r} := \pi^{q+r+s}_{q+r}(\mathcal{R}_{q+r+s}) \subseteq \mathcal{R}_{q+r}, \qquad r, s \in \mathbb{Z}_{\geq 0}$$

is called *projection*. $\mathcal{R}_q$ is called *formally integrable* if $\mathcal{R}_{q+r}$ is a fibre bundle and the projections $\pi^{q+r+s}_{q+r} : \mathcal{R}_{q+r+s} \to \mathcal{R}_{q+r}$ are surjective submersions for all $r, s \in \mathbb{Z}_{\geq 0}$.

An effective criterion to decide the formal integrability of a system of PDE was given by Goldschmidt [3]. It reduces the infinite number of conditions to a single one, once the *symbol* $g_q = (S^q T^* X \otimes V(\mathcal{E})) \cap V(\mathcal{R}_q)$ is 2-acyclic. For an introduction to symbols and Spencer cohomology see [3], [5], [10] or [9, ch. 7.2] for details in the case of jet groupoids.

Theorem 4.2 ([3, Thm 8.1])**.** *Let $\mathcal{R}_q \subseteq J_q(\mathcal{E})$ be a system of order q on $\mathcal{E}$, such that $\mathcal{R}_{q+r}$ is a subbundle of $J_{q+r}(\mathcal{E})$. If the symbol g_q is 2-acyclic, $g_{q+1} \to \mathcal{R}_q$ is a vector bundle and if the map $\pi^{q+1}_q : \mathcal{R}_{q+1} \to \mathcal{R}_q$ is surjective, then $\mathcal{R}_q$ is formally integrable.*

We will now derive the bundles and sections that describe the prolongation $\mathcal{R}_{q+r}(\omega)$ and the projection $\mathcal{R}^{(1)}_q(\omega)$ of a jet groupoid $\mathcal{R}_q(\omega)$. For a short notation, all maps to the base X (as s, t or π) will keep their name after prolongating or taking jet bundles. The prolongation $\mathcal{R}_{q+r}(\omega)$ has an obvious description:

Proposition 4.3. Let $\mathcal{F}$ be a natural bundle of order q with a section ω. Then $J_r(\mathcal{F})$ is a natural bundle of order $q + r$ and the prolongation $\mathcal{R}_{q+r}(\omega)$ of $\mathcal{R}_q(\omega)$ is the symmetry groupoid $\mathrm{Stab}^{q+r}_{J_r(\mathcal{F})}(j_r(\omega))$.

Proof. Apply the functor $J_r()$ to the Π_q-action on $\mathcal{F}$ and use the natural embedding $\Pi_{q+r} \hookrightarrow J_r(\Pi_q)$ to establish the Π_{q+r}-action on $J_r(\mathcal{F})$. Note that the image of this embedding is $J_r(\Pi_q) \cap \Pi_{q+r}$. As $\mathcal{R}_q(\omega)$ is defined as $\ker_\omega(\Phi_\omega)$, the exact sequence for $J_r(\mathcal{R}_q(\omega))$ is:

$$\begin{tikzcd}
0 \arrow[r] & J_r(\mathcal{R}_q(\omega)) \arrow[r] & J_r(\Pi_q) \arrow[r, shift left, "j_r(\Phi_\omega)"] \arrow[r, shift right, "j_r(\omega)\circ s"'] & J_r(\mathcal{F}) \\
0 \arrow[r] & \mathrm{Stab}^{q+r}_{J_r(\mathcal{F})}(j_r(\omega)) \arrow[r] \arrow[u, hook] & \Pi_{q+r} \arrow[r, shift left] \arrow[r, shift right] \arrow[u, hook] & J_r(\mathcal{F}) \arrow[u, equal]
\end{tikzcd}$$

and the intersection $J_r(\mathcal{R}_q(\omega)) \cap \Pi_{q+r}$ actually is the symmetry groupoid of $j_r(\omega)$. $\square$

The fibres of $J_r(\mathcal{F})$ are not necessarily homogeneous, so we cannot assure that $\mathcal{R}_{q+r}(\omega)$ is still a subbundle of Π_{q+r} or equivalently a Lie groupoid. However if the rank of $j_r(\Phi_\omega)$ is constant, $\mathcal{R}_{q+r}(\omega)$ is a Lie groupoid again.

To describe the projections $\mathcal{R}_q^{(1)}(\omega)$ we write $J_1(\mathcal{F})$ as a bundle associated to P_{q+1} with fibre $J_1(F) := J_1(\mathcal{F})_{y_0}$. The idea to use fibre $F_1 := J_1(F)/K_{q+1}$ to obtain the associated bundle $\mathcal{F}_1 := P_{q+1} \times_{\mathrm{GL}_{q+1}} F_1$ is due to Barakat.

Proposition 4.4. $\mathcal{F}_1 \cong P_q \times_{\mathrm{GL}_q} F_1$ is a natural bundle of order q and if $I : J_1(\mathcal{F}) \to \mathcal{F}_1$ is the projection, $\mathcal{R}_q^{(1)}(\omega)$ is the symmetry groupoid $\mathrm{Stab}^q_{\mathcal{F}_1}(I(j_1(\omega)))$.

Proof. By construction of K_{q+1}, the GL_{q+1}-action on the fibre F_1 factors over GL_q and $P_{q+1}/K_{q+1} \cong P_q$ ensures that $\mathcal{F}_1$ is a natural bundle of order q. The preimage $I^{-1}(v)$ of $v \in \mathcal{F}_{1y}$ can be written as $u_1 K_{q+1}$ with $u_1 \in I^{-1}(v)$ and $K_{q+1} \trianglelefteq \Pi_{q+1}(y,y)$. The action on $\mathcal{F}_1$ is defined by $v f_q = u_1 K_{q+1} f_{q+1} = u_1 f_{q+1} K_{q+1}$ and we have the exact sequence for $\mathrm{Stab}^q_{\mathcal{F}_1}(I(j_1(\omega)))$:

$$0 \longrightarrow \mathrm{Stab}^q_{\mathcal{F}_1}(I(j_1(\omega))) \longrightarrow \Pi_q \underset{I(j_1(\omega))\circ s}{\overset{I(j_1(\Phi_\omega))}{\rightrightarrows}} \mathcal{F}_1.$$

The symmetry condition

$$I(j_1(\omega))(y) f_q = j_1(\omega)(y) f_{q+1} K_{q+1} \overset{!}{=} j_1(\omega)(x) K_{q+1}$$

is equivalent to the existence of a preimage $r_{q+1} \in \mathcal{R}_{q+1}(\omega)$ projecting onto f_q. $\square$

We now come to the main result of this article, which is a conceptional proof using groupoids of a theorem implicitly present in [13] and formulated by Pommaret.

Theorem 4.5. *The projection $\pi^{q+1}_q : \mathcal{R}_{q+1}(\omega) \to \mathcal{R}_q(\omega)$ is an epimorphism if and only if there is a Π_q-equivariant section $c : \mathcal{F} \to \mathcal{F}_1$, $c(u f_q) = c(u) f_q$, such that $I(j_1(\omega)) = c(\omega)$. This gives the exact sequence:*

$$0 \longrightarrow \mathcal{R}_q(\omega) \longrightarrow \Pi_q \underset{\omega\circ s}{\overset{\Phi_\omega}{\rightrightarrows}} \mathcal{F} \underset{c}{\overset{I\circ j_1}{\rightrightarrows}} \mathcal{F}_1.$$

Proof. Whenever we define an element a_q, a_{q+1} denotes an arbitrary preimage under the appropriate projection π^{q+1}_q. First assume the existence of $r_{q+1} \in (\pi^{q+1}_q)^{-1}(r_q)$ for all $r_q \in \mathcal{R}_q(\omega)$. To construct an equivariant section, we define

$$c(\omega(y)) := j_1(\omega)(y) K_{q+1}.$$

For $\omega(y) \neq u \in F_y$ there is a $g_q \in \mathrm{GL}_q(\mathbb{R}^n)$ with $u = \omega(y) g_q$, we set

$$c(u) := j_1(\omega)(y) g_{q+1} K_{q+1},$$

which is well-define due to $g_{q+1} K_{q+1}$ being the whole preimage in $\mathrm{GL}_{q+1}(\mathbb{R}^n)$. For each $f_q \in \Pi_q$, we can find $h_q \in \mathrm{GL}_q(\mathbb{R}^n)$ with

$$\omega(x) h_q = u\, f_q = \omega(y) g_q f_q = \omega(y) r_q h_q \quad \text{and} \quad f_q = g_q^{-1} r_q h_q.$$

where the existence of r_{q+1} implies the equivariance:

$$\begin{aligned} c(u\, f_q) &= j_1(\omega)(x) h_{q+1} K_{q+1} \\ &= j_1(\omega)(y) g_{q+1} (g_{q+1}^{-1} r_{q+1} h_{q+1}) K_{q+1} \\ &= c(u) f_{q+1} K_{q+1} = c(u) f_q \end{aligned}$$

Using the equivariance of c on $c(\omega(y)\, r_q) = c(\omega(y)) r_q$, we obtain

$$j_1(\omega)(y) \bar{r}_{q+1} K_{q+1} = j_1(\omega)(x) K_{q+1}$$

for an arbitrary preimage $\bar{r}_{q+1}$. There is a k_{q+1} such that $r_{q+1} = \bar{r}_{q+1} k_{q+1}$ satisfying

$$j_1(\omega)(y) r_{q+1} = j_1(\omega)(x)$$

which provides a preimage $r_{q+1} \in \mathcal{R}_{q+1}(\omega)$ for r_q. $\square$

Using the GL_q-action on the fibre F_1, all possibilities for equivariant sections c can be calculated. The resulting integrability conditions $I(j_1(\omega)) = c(\omega)$ are called *Vessiot structure equations*. They express the condition that each defining equation for $\mathcal{R}_{q+1}(\omega)$ where the jets of order $q+1$ can be eliminated must be a consequence of the equations for $\mathcal{R}_q(\omega)$.

If the Vessiot structure equations are fulfilled for a section ω, $\mathcal{R}_{q+1}(\omega)$ is transitive and a subbundle of Π_{q+1}. Then by [9], the symbol g_{q+1} is a vector bundle and we can apply Theorem 4.2 which implies formal integrability. Theorem 4.5 can be extended to non-homogeneous fibres F as long as the section ω defines a Lie groupoid.

Starting from an arbitrary transitive jet groupoid $\mathcal{R}_q$, we have found a natural bundle $\mathcal{F}$ of geometric objects and a special object ω_0, such that $\mathcal{R}_q = \mathrm{Stab}^q_{\mathcal{F}}(\omega_0)$. It has been shown that the section $j_r(\omega)$ of $J_r(\mathcal{F})$ defines the r-th prolongation $\mathcal{R}_{q+r}(\omega)$ and that $I(j_1(\omega))$ on $\mathcal{F}_1$ corresponds to the projection $\mathcal{R}_q^{(1)}(\omega)$. Based on Theorem 4.2, the projection theorem leads to a check of formal integrability directly on the level of sections ω of $\mathcal{F}$. In most cases, the integrability conditions have an immediate geometric interpretation as in the following example.

5. Example

The following calculation is due to Barakat using the Maple package `jets` [1], which contains routines for jet groupoids and natural bundles. It will be used to show explicit examples of the objects in the theoretical part.

```
>  with(jets):
```

Dimension of the base manifold X and some coordinates:

```
>  n:=2:
>  ivar:=[x1,x2]:  dvar:=[y1,y2]:
>  Ivar:=[phi1,phi2]:  Dvar:=[xi1,xi2]:
```

The jet groupoid expressing the invariance of the flat euclidean metric g on X:

```
>  (Jac,g):=(matrix(n,n,jetcoor(1,ivar,dvar)), linalg[diag](1$n));
```

$$Jac,\, g := \begin{bmatrix} y1_{x1} & y1_{x2} \\ y2_{x1} & y2_{x2} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
>  J_:=evalm(linalg[transpose](Jac) &* g &* Jac);
```

$$J\_ := \begin{bmatrix} {y1_{x1}}^2 + {y2_{x1}}^2 & y1_{x1}\, y1_{x2} + y2_{x1}\, y2_{x2} \\ y1_{x1}\, y1_{x2} + y2_{x1}\, y2_{x2} & {y1_{x2}}^2 + {y2_{x2}}^2 \end{bmatrix}$$

```
>  GR_g:=[ J_[1,1]=1, J_[1,2]=0, J_[2,2]=1];
```

$$GR\_g := [{y1_{x1}}^2 + {y2_{x1}}^2 = 1,\ y1_{x1}\, y1_{x2} + y2_{x1}\, y2_{x2} = 0,\ {y1_{x2}}^2 + {y2_{x2}}^2 = 1]$$

These equations locally define a transitive groupoid $\mathcal{R}_1(g) \subset \Pi_1$ with isotropy groups $O_2(\mathbb{R})$. They have been constructed by the action of $\mathrm{GL}_1 \cong \mathrm{GL}(\mathbb{R}^2)$ on the space F of scalar products on $\mathbb{R}^2$. So we start with the natural bundle $\mathcal{F}_g = P_1 \times_{\mathrm{GL}_1} F \cong S^2T^*X_{\geq 0}$ of symmetric positive definite 2-forms and the equations for $\mathcal{R}_1(g)$ are already in Lie form (see section 3 for P_q and GL_q). Define coordinates for $\mathcal{F}_g$ and a section ω:

```
>  uvar_g:=[u11,u12,u22]:  wvar_g:=[omega11,omega12,omega22]:
```

As in [13], the coordinate changes of $\mathcal{F}_g$ are given in the form

$$\hat{x} = \phi(x), \quad u = \Psi(\hat{x} = \phi(x), \hat{u}, \phi_q(x))$$

(mind the hats in the second equation). For shorter output, jet notation is used for $\phi(x)$ and its derivatives:

```
>  inv_g:=ezip(uvar_g,map(lhs,GR_g)):
>  F_g:=natfin(inv_g,ivar,dvar,uvar_g,Ivar,""):
>  eqn2ind(F_g,ivar,Ivar);
```

$$\begin{aligned}
&[x1 = \phi 1,\ x2 = \phi 2,\\
&u11 = {\phi 1_{x1}}^2\, u11 + 2\, \phi 1_{x1}\, \phi 2_{x1}\, u12 + {\phi 2_{x1}}^2\, u22,\\
&u12 = \phi 1_{x2}\, \phi 1_{x1}\, u11 + \phi 1_{x2}\, \phi 2_{x1}\, u12 + \phi 2_{x2}\, \phi 1_{x1}\, u12 + \phi 2_{x2}\, \phi 2_{x1}\, u22,\\
&u22 = {\phi 1_{x2}}^2\, u11 + 2\, \phi 1_{x2}\, \phi 2_{x2}\, u12 + {\phi 2_{x2}}^2\, u22]
\end{aligned}$$

The groupoid $\mathcal{R}_1(\omega)$ for a general section in Lie form:

```
>  LieFormG(F_g,ivar,dvar,Ivar,wvar_g);
```

$$\begin{aligned}
&[{y1_{x1}}^2\, \omega 11(y1,\ y2) + 2\, y1_{x1}\, y2_{x1}\, \omega 12(y1,\ y2) + {y2_{x1}}^2\, \omega 22(y1,\ y2) = \omega 11(x1,\ x2),\\
&y1_{x2}\, y1_{x1}\, \omega 11(y1,\ y2) + y1_{x2}\, y2_{x1}\, \omega 12(y1,\ y2) + y2_{x2}\, y1_{x1}\, \omega 12(y1,\ y2)\\
&+ y2_{x2}\, y2_{x1}\, \omega 22(y1,\ y2) = \omega 12(x1,\ x2),\\
&{y1_{x2}}^2\, \omega 11(y1,\ y2) + 2\, y1_{x2}\, y2_{x2}\, \omega 12(y1,\ y2) + {y2_{x2}}^2\, \omega 22(y1,\ y2) = \omega 22(x1,\ x2)]
\end{aligned}$$

The special section ω_0 for the flat metric g:

```
>  omega0:=map(rhs,GR_g);
```

$$\omega 0 := [1,\ 0,\ 1]$$

The application of Theorem 4.5 at this point gives no integrability conditions, although an arbitrary metric should not be integrable. The reason is that the symbol of $\mathcal{R}_1(\omega)$ is not yet 2-acyclic, but $\mathcal{R}_2(\omega)$ has 2-acyclic symbol. We could go on with $J_1(\mathcal{F}_g)$, but in order to keep geometrical interpretation (and short expressions) we also model the Christoffel symbols by plugging the derivatives of the transformed flat metric (GR_g) into:

$$\Gamma^k_{ij}(x) = \frac{1}{2} g^{k\mu}(x) \left(\frac{\partial g_{i\mu}}{\partial x^j}(x) + \frac{\partial g_{j\mu}}{\partial x^i}(x) - \frac{\partial g_{ij}}{\partial x^\mu}(x) \right), \tag{5.1}$$

which gives the equations for the Christoffel symbols of the flat metric in Lie form:

```
>  dJac := linalg[det](Jac):
>  Phi_Gamma := [
>  (y2[x2]*y1[x1,x1]-y2[x1,x1]*y1[x2])/dJac,
>  (y2[x2]*y1[x1,x2]-y2[x1,x2]*y1[x2])/dJac,
>  (y2[x1,x1]*y1[x1]-y1[x1,x1]*y2[x1])/dJac,
>  (y2[x1,x2]*y1[x1]-y1[x1,x2]*y2[x1])/dJac,
>  (y2[x2]*y1[x2,x2]-y2[x2,x2]*y1[x2])/dJac,
>  (y2[x2,x2]*y1[x1]-y1[x2,x2]*y2[x1])/dJac]:
```

The coordinates for the Christoffel symbols (uijk stands for Γ^i_{jk}):

```
>  uvar_Gamma:=[u111,u112,u211,u212,u122,u222]:
```

Calculate the natural bundle $\mathcal{F}_\Gamma$ of Christoffel symbols:

```
>  inv_Gamma := ezip(uvar_Gamma,Phi_Gamma):
>  F_Gamma:=natfin(inv_Gamma,ivar,dvar,uvar_Gamma,Ivar,""):
```

The result is $\mathcal{F} = \mathcal{F}_g \times_X \mathcal{F}_\Gamma \cong J_1(\mathcal{F}_g)$. Usually, $J_1(\mathcal{F})$ is only an affine bundle over $\mathcal{F}$ and does not split. The fibre F is a homogeneous GL_2-space, so each section on $\mathcal{F}$ defines a Lie groupoid.

```
> uvar:=[op(uvar_g),op(uvar_Gamma)]:
> F:=[op(F_g),op(F_Gamma[n+1..-1])]:
```

To calculate the projection to the bundle $\mathcal{F}_1$, the vector fields of infinitesimal transformations of $\mathcal{F}$. If $\xi^i(x)\frac{\partial}{\partial x^i}$ is a vector field on X, it can be extended to $\mathcal{F}$:

```
> vec:=natfin2inf(F,ivar,Ivar,Dvar,"");
```

$$
\begin{aligned}
&vec := [[\xi 1, [x1]], [\xi 2, [x2]], [-2\, u11\, \xi 1_{x1} - 2\, u12\, \xi 2_{x1}, [u11]],\\
&[-\xi 1_{x1}\, u12 - \xi 1_{x2}\, u11 - \xi 2_{x1}\, u22 - \xi 2_{x2}\, u12, [u12]],\\
&[-2\, u12\, \xi 1_{x2} - 2\, u22\, \xi 2_{x2}, [u22]],\\
&[-\xi 1_{x1,\, x1} - \xi 1_{x1}\, u111 + \xi 1_{x2}\, u211 - 2\, \xi 2_{x1}\, u112, [u111]],\\
&[-\xi 1_{x1,\, x2} - \xi 1_{x2}\, u111 + \xi 1_{x2}\, u212 - \xi 2_{x1}\, u122 - \xi 2_{x2}\, u112, [u112]],\\
&[-2\, \xi 1_{x1}\, u211 - \xi 2_{x1,\, x1} + \xi 2_{x1}\, u111 - 2\, \xi 2_{x1}\, u212 + \xi 2_{x2}\, u211, [u211]],\\
&[-\xi 1_{x1}\, u212 - \xi 1_{x2}\, u211 - \xi 2_{x1,\, x2} + \xi 2_{x1}\, u112 - \xi 2_{x1}\, u222, [u212]],\\
&[-\xi 1_{x2,\, x2} + \xi 1_{x1}\, u122 - 2\, \xi 1_{x2}\, u112 + \xi 1_{x2}\, u222 - 2\, \xi 2_{x2}\, u122, [u122]],\\
&[-2\, \xi 1_{x2}\, u212 - \xi 2_{x2,\, x2} + \xi 2_{x1}\, u122 - \xi 2_{x2}\, u222, [u222]]]
\end{aligned}
$$

The above list denotes a vector field. It is read as follows: $[\xi 1, [x1]]$ stands for $\xi^1(x)\frac{\partial}{\partial x^1}$ and the complete vector field is obtained by adding up all list entries. Each choice of $\xi^i, \ldots, \xi^i_{x^i,x^j}$ gives an infinitesimal transformation of $\mathcal{F}$. We calculate the coordinates of $\mathcal{F}_1$ that express the projection $I : J_1(\mathcal{F}) \to \mathcal{F}_1$:

```
> F1:=F1coor(vec,ivar,Dvar,uvar);
```

$$
\begin{aligned}
F1 := [&u11_{x1},\, u11_{x2},\, u12_{x1},\, u12_{x2},\, u22_{x1},\, u22_{x2},\\
&u111_{x2} - u112_{x1},\, u112_{x2} - u122_{x1}, u211_{x2} - u212_{x1},\, u212_{x2} - u222_{x1}]
\end{aligned}
$$

```
> d1:=nops(F1):
> vvar := [v1,v2,v3,v4,v5,v6,v7,v8,v9,v10]:
```

All further computations only need the infinitesimal coordinate changes of $\mathcal{F}_1$. Setting zero order jets $\xi^i = 0$ to zero and collecting for higher order jets of ξ^i, the list L1 contains a representation of the Lie algebra of $\mathrm{GL}_2(\mathbb{R}^2)$ as vertical vector fields on $\mathcal{F}_1$.

```
> inv1 := ezip(vvar,F1):
> vec1:=natinfG(vec,inv1,ivar,uvar,vvar,Dvar):
> L1:=lstvec(sortcon(vec1,[op(uvar),op(vvar)]),ivar,Dvar,""):
```

Before calculating the possible equivariant sections of $\mathcal{F}_1$, we will modify the coordinates of $\mathcal{F}_1$ to obtain a vector bundle atlas. This is achieved by choosing the coordinates for the fibres of $\mathcal{F}_1 \to \mathcal{F}$ to be K_2-invariant (see sequence (3.1) for K_{q+1}):

```
> cvar := [c1,c2,c3,c4,c5,c6,c7,c8,c9,c10]:
> subv:=map(a->vvar[a]=cvar[a](op(uvar)),[$1..nops(cvar)]):
> cndi:=map(i->subs(subv,invcond(L1[1][n^2+1..-1],
> [lhs(subv[i])-rhs(subv[i])],L1[2])[1]),[$1..d1]):
> sol_Gamma := map(ci->jsolve(cndi[ci], uvar,
> [cvar[ci](op(uvar))],""), [$1..d1]):
```

The results all depend on arbitrary functions $\_F1(u11, u12, u22)$, which will be set to zero:

```
> sol_Gamma[1];
```

$$[\mathrm{c1}(u11,\, u12,\, u22,\, u111,\, u112,\, u211,\, u212,\, u122,\, u222) = 2\, u11\, u111 + 2\, u12\, u211 + \_\mathrm{F1}(u11,\, u12,\, u22)]$$

```
> sol_Gamma := eval(map(a->op(subs(_F1=0,a)),sol_Gamma)):
```

The new infinitesimal coordinate changes show the vector bundle structure of $\mathcal{F}_1 \to \mathcal{F}$:

```
> inv1_1 := zip((a,b)->lhs(a) = rhs(a) - rhs(b),inv1,sol_Gamma):
> vec1_1:=natinfG(vec,inv1_1,ivar,uvar,vvar,Dvar);
> L1_1:=lstvec(sortcon(vec1_1,[op(uvar),op(vvar)]),ivar,Dvar,""):
```

$$\begin{aligned}
&vec1_1 := [\ldots vec \ldots \\
&[-3\, v1\, \xi 1_{x1} - v2\, \xi 2_{x1} - 2\, v3\, \xi 2_{x1},\, [v1]], \\
&[-v1\, \xi 1_{x2} - v2\, \xi 2_{x2} - 2\, v2\, \xi 1_{x1} - 2\, v4\, \xi 2_{x1},\, [v2]], \\
&[-v1\, \xi 1_{x2} - \xi 2_{x2}\, v3 - 2\, v3\, \xi 1_{x1} - v4\, \xi 2_{x1} - \xi 2_{x1}\, v5,\, [v3]], \\
&[-\xi 1_{x2}\, v2 - v3\, \xi 1_{x2} - 2\, v4\, \xi 2_{x2} - \xi 1_{x1}\, v4 - v6\, \xi 2_{x1},\, [v4]], \\
&[-2\, v3\, \xi 1_{x2} - 2\, v5\, \xi 2_{x2} - v5\, \xi 1_{x1} - v6\, \xi 2_{x1},\, [v5]], \\
&[-2\, v4\, \xi 1_{x2} - v5\, \xi 1_{x2} - 3\, v6\, \xi 2_{x2},\, [v6]], \\
&[-\xi 2_{x2}\, v7 - \xi 1_{x1}\, v7 - \xi 2_{x1}\, v8 + \xi 1_{x2}\, v9,\, [v7]], \\
&[-\xi 1_{x2}\, v7 - 2\, \xi 2_{x2}\, v8 + \xi 1_{x2}\, v10,\, [v8]], \\
&[\xi 2_{x1}\, v7 - 2\, \xi 1_{x1}\, v9 - \xi 2_{x1}\, v10,\, [v9]], \\
&[\xi 2_{x1}\, v8 - \xi 1_{x2}\, v9 - \xi 2_{x2}\, v10 - \xi 1_{x1}\, v10,\, [v10]]]
\end{aligned}$$

We are now able to calculate all equivariant sections c. The infinitesimal conditions for equivariance are obtained by applying all vector fields of the Lie algebra of GL_2 to $v_i - c_i(u) = 0$ and then substituting $v_i \to c_i$. The same method was used for the K_2-invariance.

```
> subv_1:=map(a->vvar[a]=cvar[a](op(uvar)),[$1..nops(cvar)]):
> cnd1 := subs(subv_1,invcond(L1_1[1],map(a->lhs(a)-rhs(a),subv_1),
> L1_1[2])[1]):
> cc := jsolve(cnd1,uvar,map(a->a(op(uvar)),cvar),""):
```

The Vessiot structure equations show the integrability conditions with the equivariant sections on the right hand side:

```
> Ves := subs(inv1_1,subs(cc,subv_1));
```

$$
\begin{aligned}
&Ves := [u11_{x1} - 2\,u11\,u111 - 2\,u12\,u211 = 0,\\
&u11_{x2} - 2\,u11\,u112 - 2\,u12\,u212 = 0,\\
&u12_{x1} - (u111 + u212)\,u12 - u11\,u112 - u22\,u211 = 0,\\
&u12_{x2} - (u112 + u222)\,u12 - u22\,u212 - u11\,u122 = 0,\\
&u22_{x1} - 2\,u12\,u112 - 2\,u22\,u212 = 0,\\
&u22_{x2} - 2\,u12\,u122 - 2\,u22\,u222 = 0,\\
&u111_{x2} - u112_{x1} - u212\,u112 + u122\,u211\\
&\quad = \_C1\,u12 + \sqrt{-u12^2 + u22\,u11}\,\_C2,\\
&u112_{x2} - u122_{x1} - (u111 - u212)\,u122 - u112\,u222 + u112^2 = \_C1\,u22,\\
&u211_{x2} - u212_{x1} - u212^2 + u212\,u111 - (u112 - u222)\,u211 = -u11\,\_C1,\\
&u212_{x2} - u222_{x1} - u122\,u211 + u212\,u112\\
&\quad = -\_C1\,u12 + \sqrt{-u12^2 + u22\,u11}\,\_C2]
\end{aligned}
$$

They show that all equivariant sections c can be parametrised by two constants ($\_C1$ and $\_C2$). The second constant $\_C2$ is special to the 2-dimensional case and we obtain $\_C2 = 0$ using the Jacobi conditions in [9, Thm. 7.4.8]. The first six integrability conditions express the Christoffel symbols in terms of the metric and its first order derivatives (cf. eq. (5.1)):

```
> nrsolve(Ves[1..6],uvar_Gamma)[1];
```

$$
\begin{aligned}
&[u111 = -\frac{1}{2}\,\frac{-u11_{x1}\,u22 + 2\,u12\,u12_{x1} - u12\,u11_{x2}}{-u12^2 + u22\,u11},\\
&u112 = -\frac{1}{2}\,\frac{-u11_{x2}\,u22 + u12\,u22_{x1}}{-u12^2 + u22\,u11},\\
&u211 = \frac{1}{2}\,\frac{2\,u11\,u12_{x1} - u12\,u11_{x1} - u11\,u11_{x2}}{-u12^2 + u22\,u11},\\
&u212 = \frac{1}{2}\,\frac{u11\,u22_{x1} - u12\,u11_{x2}}{-u12^2 + u22\,u11},\\
&u122 = -\frac{1}{2}\,\frac{u22_{x2}\,u12 - 2\,u22\,u12_{x2} + u22\,u22_{x1}}{-u12^2 + u22\,u11},\\
&u222 = \frac{1}{2}\,\frac{u11\,u22_{x2} - 2\,u12\,u12_{x2} + u12\,u22_{x1}}{-u12^2 + u22\,u11}]
\end{aligned}
$$

Starting with a metric, they are always fulfilled, but an arbitrary section of $\mathcal{F}$ allows to choose metric and Christoffel symbols independently. The last four integrability conditions express components of the Riemann curvature tensor as derivatives of the Christoffel symbols. The equations are equivalent ($\_C2 = 0$) to the condition of a metric with constant scalar curvature:

$$R^k_{lij} = \partial_i \Gamma^k_{lj} - \partial_j \Gamma^k_{li} + \Gamma^r_{lj}\,\Gamma^k_{ri} - \Gamma^r_{li}\,\Gamma^k_{rj} = \_C1(\delta^k_j g_{li} - \delta^k_i g_{lj}).$$

The calculations with `jets` complement the the theory with explicit coordinate changes of the natural bundles $\mathcal{F}$ and $\mathcal{F}_1$, which are equivalent to the Π_q-action on $\mathcal{F}$. Vessiot's structure equations can now be used to check the integrability of jet groupoids.

In [13], the structure equations are solved for the constants, which is an alternative choice of coordinates for $\mathcal{F}_1$. Usually, this leads to larger expressions and hides the geometrical interpretation. If the typical fibre of $\mathcal{F}$ is not homogeneous, the freedom for equivariant sections may extend from constants to smooth invariants.

Vessiot's structure equations can also be applied to test whether two geometric objects are formally equivalent, which is connected to the integrability of the corresponding groupoids an equivariant sections.

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